Homework 1 Solutions

1. Let m, n, p be distinct positive integers and $m > 1$.

   - Let A be an $m \times n$ matrix
   - Let B and C be $n \times p$ matrices
   - Let D be a $p \times m$ matrix
   - Let $\mathbf{v} \in \mathbb{R}^n$

   For each of the following expressions, describe what kind of object is produced (including the dimension if the object is a vector or the number of rows and columns if the object is a matrix), or if the expression is not well- defined, explain why.

   (a) $A\mathbf{v}$

   This is an $m-$dimensional vector.

   (b) $B\mathbf{v}$

   This is undefined, since B has p rows and $\mathbf{v}$ has n entries.

   (c) $\mathbf{v}A$

   This is undefined, since $\mathbf{v}$ has one column and A has $m > 1$ rows.

   (d) AB

   This is an $m \times p$ matrix.

   (e) ABC

   This is undefined. Since B has p columns and C has n rows, the product BC is undefined and hence $ABC = A(BC)$ is undefined.

   (f) ABD

   This is an $m \times m$ matrix.

   (g) $A + B$

   This is undefined, since A and B have different sizes.

(h) $A(B+C)$

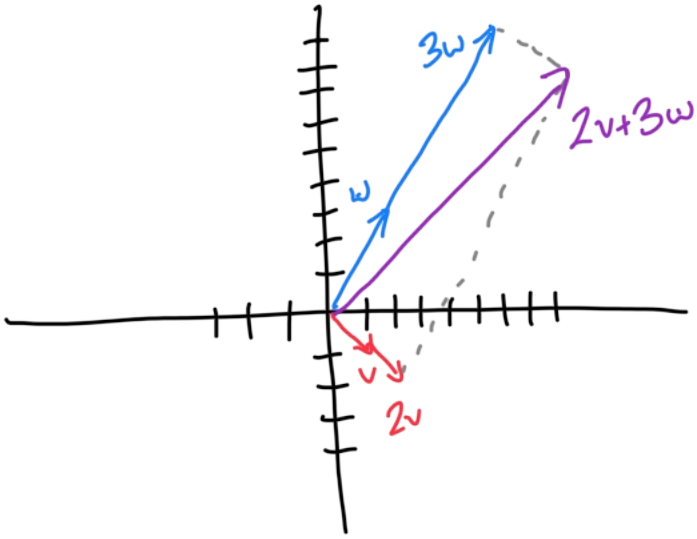
This is an $m \times p$ matrix.

2. Consider the vectors $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(a) Compute the linear combination $2\mathbf{v} + 3\mathbf{w}$.

$$2\mathbf{v} + 3\mathbf{w} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

(b) Sketch $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$. Then sketch the linear combination vector $2\mathbf{v} + 3\mathbf{w}$ by first scaling $\mathbf{v}$ and $\mathbf{w}$, and then using the parallelogram rule.



(c) Set up and solve a system of equations to answer the following question:
Is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ a linear combination of $\mathbf{v}$ and $\mathbf{w}$? If so, what are the weights?

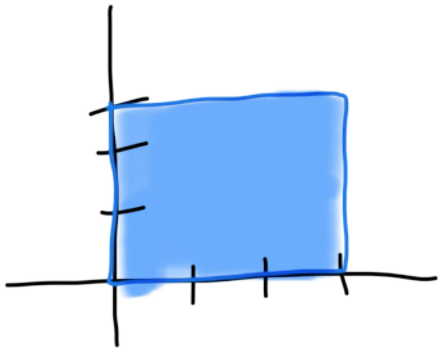
We need to find weights $x, y \in \mathbb{R}$ such that $x\mathbf{v} + y\mathbf{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Plugging in $\mathbf{v}$ and $\mathbf{w}$, we have:

$$x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} x + 2y \\ -x + 3y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Hence we must solve the system: $\begin{matrix} x+2y=3 \\ -x+3y=4 \end{matrix}$ Adding the two equations yields $5y=7$ or $y=\frac{7}{5}$. Plugging this back into the first equation gives $x=\frac{1}{5}$. Since this system has a solution, $\begin{bmatrix}3\\4\end{bmatrix}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ with weights $\frac{1}{5}$ and $y=\frac{7}{5}$.

3. Consider the linear transformation T given by the 2×2 matrix $A=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}$.

   (a) Plot the vectors $\begin{bmatrix}a\\b\end{bmatrix}$ where $0\le a\le 3$ and $0\le b\le 3$ in the domain $\mathbb{R}^2$. Here, instead of drawing the arrows, shade in the region populated by the vectors.

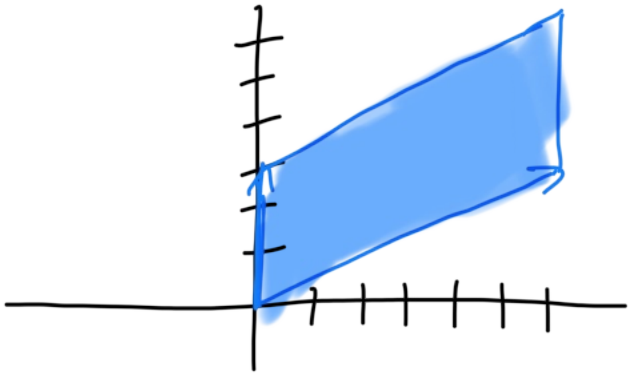


   Note that $\begin{bmatrix}a\\b\end{bmatrix}$ with $0\le a,b\le 3$ can be expressed as a linear combination of $\begin{bmatrix}3\\0\end{bmatrix}$ and $\begin{bmatrix}0\\3\end{bmatrix}$ with weights between 0 and 1; namely $\begin{bmatrix}a\\b\end{bmatrix}=\frac{a}{3}\begin{bmatrix}3\\0\end{bmatrix}+\frac{b}{3}\begin{bmatrix}0\\3\end{bmatrix}$. Hence the region sketched above, which contains all such vectors, is region inside the parallelogram formed by the vectors $\begin{bmatrix}3\\0\end{bmatrix}$ and $\begin{bmatrix}0\\3\end{bmatrix}$. This information was not necessary for the above plot, but it is useful in understanding how to draw the next plot.

   (b) Plot the vectors $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$ where $0\le a\le 3$ and $0\le b\le 3$ in the codomain $\mathbb{R}^2$. Here, instead of drawing the arrows, shade in the region populated by the vectors.

   Since T is a linear transformation, $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)=\frac{a}{3}T\left(\begin{bmatrix}3\\0\end{bmatrix}\right)+\frac{b}{3}T\left(\begin{bmatrix}0\\3\end{bmatrix}\right)$; that is the output vector $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$ can be expressed as a linear combination of $T\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$ with weights between 0 and 1. If follows that the region populated by all

such vectors is the region inside parallelogram formed by $T\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$ and $T\left(\begin{bmatrix}3\\0\end{bmatrix}\right)$, which is drawn below.



(c) Let S be the linear transformation given by $B = \begin{bmatrix}-1 & 1 & 2\\ 2 & 1 & -1\end{bmatrix}$. Compute the matrices associated to the linear transformations $S \circ T$ and $T \circ S$ (if they exist).

The matrix associated to $T \circ S$ is

$$AB = \begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\\ 2 & 1 & -1\end{bmatrix} = \begin{bmatrix}-2 & 2 & 4\\ 1 & 2 & 1\end{bmatrix}$$

Since $S : \mathbb{R}^3 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$, the composition $S \circ T$ is undefined; indeed the product BA is also undefined.

4. Consider the map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = mx + b$, where $m, b \in \mathbb{R}$ and $m \neq 0$.

(a) Show that if $b \neq 0$, then T is not a linear transformation.

Let $b \neq 0$. Then $T(1+1) = T(2) = 2m + b$ and $T(1) + T(1) = (m+b) + (m+b) = 2m + 2b$ Since $b \neq 0$, $2m + b \neq 2m + 2b$ and so $T(1+1) \neq T(1) + T(1)$.

(b) Show that if $b = 0$, then T is a linear transformation.

Let $b = 0$. Then $T(x) = mx$. To show it is linear, we must verify the two linear transformation properties:

- $T(x + y) = m(x + y) = mx + my = T(x) + T(y)$ for all $x, y \in \mathbb{R}$.
- $T(cx) = m(cx) = c(mx) = cT(x)$ for all $c, x \in \mathbb{R}$.

5. Explain why the following solutions sets are the same.

(a) The set of all $x, y \in \mathbb{R}$ such that $x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

(b) The set of all $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T\Big(\begin{bmatrix} x \\ y \end{bmatrix}\Big) = \begin{bmatrix} x - 2y \\ 2x + y \end{bmatrix}$.

(c) The set of all $x, y \in \mathbb{R}$ such that $x - 2y = 1$ and $2x + y = -1$.

6. Solve the following system by using Gaussian-Jordan elimination to put the associated augmented matrix in RREF:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0 \end{aligned}$$

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow[\text{Replace R2 with R2-2R1}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[\text{Replace R3 with R3+2R2}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right]$$

$$\xrightarrow[\text{Multiply R3 by } \frac{1}{8}]{} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Replace R2 by R2-3R3}]{\text{Replace R1 by R1-R3}} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Replace R1 by R1-3R2}]{} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Multiply R1 by } \frac{1}{2}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The solution is $x = 2, y = 1, z = 1$.

7. Recall the associative property for vector addition: $\mathbf{v} + (\mathbf{w} + \mathbf{u}) = (\mathbf{v} + \mathbf{w}) + \mathbf{u}$ for all $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^n$.

(a) Prove this property.

Let $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{v} + (\mathbf{w} + \mathbf{u}) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 + u_1 \\ \vdots \\ w_n + u_n \end{bmatrix} &= \begin{bmatrix} v_1 + (w_1 + u_1) \\ \vdots \\ v_n + (w_n + u_n) \end{bmatrix} \\ &= \begin{bmatrix} (v_1 + w_1) + u_1 \\ \vdots \\ (v_n + w_n) + u_n \end{bmatrix} \\ &= \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = (\mathbf{v} + \mathbf{w}) + \mathbf{u} \end{aligned}$$

(b) Symbolically prove this property using Julia for three symbolic vectors in $\mathbb{R}^5$. Take a screenshot of your code and output and add it as the last page of your solutions.